

Limit properties of global interaction stochastic quantum walks on directed graphs

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Abstract

In this paper we show that the subspace of stationary states of quantum stochastic walk in the global interaction regime for undirected graphs equals the one for original continuous quantum walk. This demonstrates that the global interaction quantum stochastic walk do not have the relaxing property. We also show that in this model an arbitrary mixed state converges to a stationary state. This cannot be generalized and we provide examples for directed graph which do not have the convergence property. The examples are presented for both models with and without the spontaneous moralization property. We utilize the stationary states of nonmoralizing global interaction quantum stochastic walk for analysing the observance of directed graph structure. We argue that this model is suitable for using directed graphs for quantum information processing.

Keywords Quantum stochastic walks, stationary states, state convergence

1 Introduction

Results from past few decades show that the choice of the quantum analogue of classical random walk is highly non-unique. Some of the most popular models include discrete coined quantum walk [1], continuous quantum walk [2], Szegedy walk [3], open walk [4], staggered quantum walk [5], and quantum stochastic walk [6]. They have found applications in designing PageRank algorithm [7, 8, 15], search algorithms [9–11], solving triangle problem [12], and describing chemical reactions [13]. All of them outperform their classical counterpart at least for some large class of graphs. From the algorithmic perspective, it is crucial to understand the limiting behaviour of the walks, because it decides about the usefulness of the model. This includes the propagation speed, which decides about efficiency of the search algorithm [14], mixing time and relaxing, which find applications in PageRank algorithm [15–17], limit theorem [18, 19], and trapping [20].

A special kind of continuous quantum evolution is stochastic walk, which generalizes both classical and quantum walk [6]. The model has been investigated in the context of relaxing property [15, 16], propagation speed [21], and applications to various physics [13] and computer science [15] problems. In particular, it has been proven that in the case of local interaction quantum stochastic walks, if the graph is undirected and connected, then the evolution has relaxing property [16], *i.e.* there exist a single mixed state ϱ_∞ , such that for arbitrary initial mixed state $\varrho(0)$, we have $\varrho(0) \rightarrow \varrho_\infty$, for $t \rightarrow \infty$. However, it has been shown, that the local

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interaction destroys the ballistic propagation [21]. Contrary, the global interaction case preserves the ballistic propagation.

The main contribution of this paper is the description of the limit properties of the global interaction quantum stochastic walks. We show that the stationary states are precisely those from the original continuous quantum walk. Moreover, we demonstrate that for an arbitrary initial mixed state $\varrho(0)$ there exist stationary state ϱ_∞ such that $\varrho(0) \rightarrow \varrho_\infty$ in time limit. Contrary, we provide examples of directed graphs and initial states for both moralizing and nonmoralizing models, for which the evolution is periodic in time. At the same time, we conjecture that in the nonmoralizing case the probability distribution of the naturally chosen measurement converge for an arbitrary initial configuration.

We also study the properties of nonmoralizing quantum stochastic walk model. For a selected directed graphs we analyse how well, depending on smoothing parameter ω , the evolution preserves the graph structure. We show that the greater the impact of the Lindbladian part is, the more probability cumulates in the close neighbourhood of sink vertices. Moreover, we conjecture that for $\omega > 0.7$ the evolution preserves the directed graph structure monotonically with ω .

The paper is organized as follows. In Sec. 2 we provide some basic definitions and recall the model of quantum stochastic walks. In Sec. 3 we prove a theorem about the form of stationary states for the undirected graphs. We also provide examples of directed graphs where the property does not hold. In Sec. 4 we analyse how global interaction quantum stochastic walks preserves the digraph structure. We conclude our results in Sec. 5.

2 Preliminaries

To define quantum stochastic walks in general, let us start with the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [22–24]

$$\frac{d}{dt}\varrho = \mathcal{M}[\varrho] = -i[H, \varrho] + \sum_{L \in \mathcal{L}} \left(L\varrho L^\dagger - \frac{1}{2}\{L^\dagger L, \varrho\} \right), \quad (1)$$

where $\{A, B\}$ is the anticommutator and \mathcal{M} is the evolution superoperator. Here H is the Hamiltonian, which describes the evolution of the closed system, and \mathcal{L} is the collection of Lindblad operators, which describes the evolution of the open system. This master equation describes general continuous evolution of mixed quantum states.

In the case where H and \mathcal{L} do not depend on time, we say that Eq. (1) describes the Markovian evolution of the system. Henceforth, we can solve the differential equation analytically: if we choose initial state $\varrho(0)$, then

$$|\varrho(t)\rangle\rangle = S_t|\varrho(0)\rangle\rangle, \quad (2)$$

where

$$S_t = \exp \left[-it (H \otimes \mathbb{1} - \mathbb{1} \otimes \bar{H}) + t \sum_{L \in \mathcal{L}} \left(L \otimes \bar{L} - \frac{1}{2} (L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^\dagger L) \right) \right], \quad (3)$$

and $|\cdot\rangle\rangle$ denotes the vectorization of the matrix (see eg. [25]).

The GKSL master equation was used for defining quantum stochastic walks. They are the generalization of both classical random walks and quantum walks [6]. Both H and \mathcal{L} correspond to the graph structure, however one may verify that at least a choice of Lindblad operators may be non-unique [6, 21]. Suppose we have a undirected graph $G = (V, E)$. Two main models can be distinguished. In both models we choose the Hamiltonian H to be the adjacency matrix of the graph. In the *local environment interaction case* each Lindblad operator corresponds to a single edge, $\mathcal{L} = \{|w\rangle\langle v| : (v, w) \in E\}$, and in the *global environment interaction case* we choose

a single Lindblad operator $\mathcal{L} = \{H\}$. Note, that in the case of $\mathcal{L} = \emptyset$ we recover the original continuous quantum walk on mixed states.

To analyse the impact of Lindbladian part, we add a smoothing parameter $\omega \in [0, 1]$

$$S_{t,\omega} = \exp \left[-i(1-\omega)t(H \otimes \mathbb{1} - \mathbb{1} \otimes \bar{H}) + \omega t \sum_{L \in \mathcal{L}} \left(L \otimes \bar{L} - \frac{1}{2} (L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^\dagger L) \right) \right], \quad (4)$$

In [21] we have shown by analysing the Hurst exponent that the local case leads to the classical propagation of the walk. Oppositely, in the case of the global interaction, the ballistic propagation was obtained for arbitrary middle value ω . Henceforth, we find the choice of the global interaction case particularly interesting and only this case will be analysed here.

Global interaction quantum stochastic walk suffers for the spontaneous moralization and the resulting process fails to reproduce the structure of the original graph. To prevent this a correction scheme based on the system enlargement has been proposed in [14]. The scheme consist of following steps: first we combine with each vertex a subspace of the system of the dimension equal to the indegree of the vertex (in the case of source vertices the subspace is onedimensional). Then we need to choose a family of orthogonal matrices, which destroys the moralization. The last thing is to add a rotating Hamiltonian which works locally on the vertices subspaces. In this model a natural measurement is the collection of operations which projects the state onto the subspace of vertex.

3 Stationary states of quantum stochastic walks

3.1 Undirected graphs

We start the section with providing the general result for the commuting operators. The result generalize the case of quantum stochastic walk with global interaction for all undirected graphs and for some directed graphs, including circulant matrices.

Theorem 1. *Suppose we consider the Eq. (2) in the case of commuting Lindbladian operators \mathcal{L} and Hamiltonian H . Then the evolution operations S_t is of the form*

$$S_{t,\omega} = (U \otimes \bar{U}) \exp(t D_{S_{t,\omega}}) (U \otimes \bar{U})^\dagger, \quad (5)$$

where

$$D_{S_{t,\omega}} = -i(1-\omega)(D_H \otimes \mathbb{1} - \mathbb{1} \otimes D_H) + \omega \sum_{L \in \mathcal{L}} (D_L \otimes \bar{D}_L - \frac{1}{2} \bar{D}_L D_L \otimes \mathbb{1} - \frac{1}{2} \mathbb{1} \otimes D_L \bar{D}_L). \quad (6)$$

Here we assume, U is an unitary operator and D_H, D_L are diagonal operators.

Proof. The proof comes directly from the eigendecompositions of the operators. Since all operators commute, it is possible to find common eigendecomposition with the same unitary matrix. By this we can easily find the result. \square

One can note the global interaction quantum stochastic walk on undirected graphs is a special case of the evolution described in the above theorem. In the walk model the difference comes from the size of \mathcal{L} , where we choose only single Lindbladian operator. Hence we prove a result concerning undirected graphs.

Theorem 2. *The stationary states in the quantum stochastic walk with global interaction on undirected graphs are precisely the stationary states in continuous quantum walk. Moreover, for $\omega \in (0, 1]$, and for arbitrary initial state $\varrho(0)$, there exists state ϱ_∞ such that $\varrho(0)$ converges to ϱ_∞ .*

Proof. By the model construction we have $\mathcal{L} = H$. Because of that the formula Eq. (6) simplifies to

$$D_{S_{t,\omega}} = -i(1 - \omega)(D \otimes \mathbb{1} - \mathbb{1} \otimes D) + \omega(D \otimes D - \frac{1}{2}D^2 \otimes \mathbb{1} - \frac{1}{2}\mathbb{1} \otimes D^2). \quad (7)$$

Here we assume that $H = UDU^\dagger$. Since H is hermitian, the D operator is real-valued diagonal matrix. The diagonal entries of operator $D_{S_{t,\omega}}$ are eigenvalues which characterize the evolution. Since

$$\begin{aligned} \langle i, j | D_{S_{t,\omega}} | i, j \rangle &= -i(1 - \omega)(\langle i | D | i \rangle - \langle j | D | j \rangle) + \omega(\langle i | D | i \rangle \langle j | D | j \rangle - \frac{1}{2}(\langle i | D^2 | i \rangle + \langle j | D^2 | j \rangle)) = \\ &= -i(1 - \omega)(\langle i | D | i \rangle - \langle j | D | j \rangle) - \frac{\omega}{2}(\langle i | D | i \rangle - \langle j | D | j \rangle)^2. \end{aligned} \quad (8)$$

Note that the imaginary part of the eigenvalue correspond to the pure Hamiltonian evolution, called the original continuous quantum walk. Since the real part equals zero if the imaginary part equals zero, we obtained the first part of the theorem. The latter part comes from the reversed implication. Convergence comes from the fact that the operator does not have any purely imaginary eigenvalues. \square

Note that the result from the above theorem implies that we can generate the stationary states from the continuous quantum walk by adding the same Lindbladian operator. In the next section we show that Theorem 2 cannot be generalized for directed graphs. We conclude this section with the following simple remark concerning the relaxing property.

Remark 3. *For an arbitrary undirected graph with more than one vertex, the global interaction stochastic quantum walk evolution does not have the relaxing property.*

The remark follows from the fact that the space of stationary states is always non-trivial.

3.2 Directed graphs

In this section we provide an example of directed graph for which we do not necessary obtain stationary state. It has been proven that the evolution converges for arbitrary initial state iff all nonzero eigenvalues of $S_{t,\omega}$ have negative real part [26]. We found an example of a digraph which do not satisfy the condition, and provide an exemplary initial state which results in periodic evolution. One should note, that it is possible (and more probable) to find quasi-periodic states.

Theorem 4. *Let us take the evolution for which the only Lindbladian operator is an adjacency matrix of the directed graph and the Hamiltonian is an adjacency matrix of the underlying graph. Then there exists a directed graph G and an initial state $\varrho(0)$ for which the evolution is periodic in time for an arbitrary value of the smoothing parameter $\omega \in (0, 1]$.*

Proof. As an example we choose a graph presented in Fig. 1. The graph and its underlying graph are circulant matrices. Therefore we can apply Eq. (6) to find out that there exists eigenvalue of the form $2(1 - \omega)i$ with corresponding eigenvector $|C_2\rangle\overline{|C_4\rangle}$, where $\overline{(\cdot)}$ denotes the element-wise conjugation and $|C_i\rangle$ is the i -th eigenvector of a circulant matrix of the form

$$|C_i\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^7 \exp\left(\frac{2\pi i j}{7}\right) |i\rangle. \quad (9)$$

We need to find a matrix which is not orthogonal to $|C_2\rangle\langle C_4|$. Our exemplary initial state is

$$\varrho(0) = \frac{1}{2}(|C_2\rangle + |C_4\rangle)(\langle C_2| + \langle C_4|) \quad (10)$$

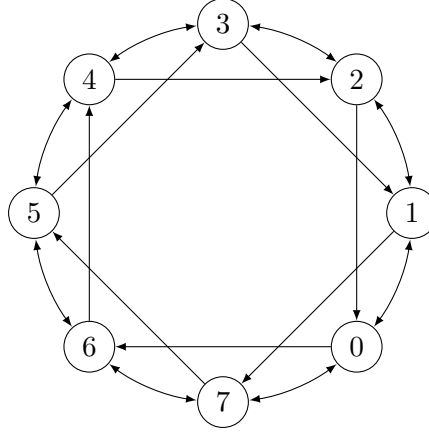


Figure 1: An example of graph, for which there exist state which does not converge

The $\varrho(t)$ takes the form

$$\varrho(t) = \frac{1}{2}(|C_2\rangle\langle C_2| + |C_4\rangle\langle C_4| + e^{2i(1-\omega)t}|C_2\rangle\langle C_4| + e^{-2i(1-\omega)t}|C_4\rangle\langle C_2|). \quad (11)$$

Since $\varrho(t)$ is periodic with period $\frac{\pi}{(1-\omega)}$, we obtain the result. \square

Note, that for different t we can obtain different state in the sense of possible measurement output. For example we have $\langle 0|\varrho(0)|0\rangle = \frac{1}{4}$, but at the same time we have $\langle 0|\varrho(\frac{\pi}{2(1-\omega)})|0\rangle = 0$.

Remark 5. *The evolution, for which the only Lindbladian operator is an adjacency matrix of the directed graph and the Hamiltonian is an adjacency matrix of the underlying graph, does not converge in general, even in the sense of the canonical measurement probability distribution.*

3.3 Nonmoralizing model

The nonmoralizing model has been introduced in [14] as a tool for modelling continuous walks on directed graphs. In this section we provide an similar example of a graph and initial state such that it does not converge. Similarly we found a digraph, for which an evolution operator $S_{t,\omega}$ has an imaginary part. Again, it is easy to find a state for which the evolution is quasi-periodic.

Theorem 6. *Let us take the nonmoralizing evolution described in [14]. Then there exists a directed graph G and initial state $\varrho(0)$ for which the evolution is periodic in time for an arbitrary value of the smoothing parameter $\omega \in (0, 1]$.*

Proof. Let us take a graph presented in Fig. 2. Using the scheme presented in [14], new graph will consist of 5 copies of vertex v_0 , two copies of vertex v_4 and v_5 , and single copy of other vertices. As the orthogonal matrices we choose the Fourier matrices and for removing the premature localization we choose the rotating Hamiltonian \tilde{H}_{rot} constructed from the Hamiltonians of the form

$$\begin{bmatrix} 0 & i & & & \\ -i & \ddots & \ddots & & \\ & \ddots & \ddots & i & \\ & & -i & 0 & \end{bmatrix}. \quad (12)$$

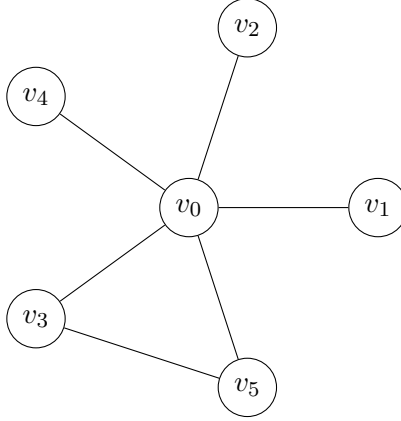


Figure 2: An example of graph for which there exist a state which does not converge in the case of non-moralizing evolution

Let us choose two eigenvectors of the rotating hamiltonian

$$|\lambda_{\tilde{H}_{\text{rot}}}^1\rangle = \frac{1}{2\sqrt{3}}|v_0^0\rangle - \frac{i}{2}|v_0^1\rangle - \frac{1}{\sqrt{3}}|v_0^2\rangle + \frac{i}{2}|v_0^3\rangle + \frac{1}{2\sqrt{3}}|v_0^4\rangle, \quad (13)$$

$$|\lambda_{\tilde{H}_{\text{rot}}}^2\rangle = \frac{1}{2\sqrt{3}}|v_0^0\rangle - \frac{i}{2}|v_0^1\rangle - \frac{1}{\sqrt{3}}|v_0^2\rangle + \frac{i}{2}|v_0^3\rangle + \frac{1}{2\sqrt{3}}|v_0^4\rangle. \quad (14)$$

$$(15)$$

One can show that the vectors $|\lambda_{\tilde{H}_{\text{rot}}}^1, \bar{\lambda}_{\tilde{H}_{\text{rot}}}^1\rangle, |\lambda_{\tilde{H}_{\text{rot}}}^1, \bar{\lambda}_{\tilde{H}_{\text{rot}}}^2\rangle, |\lambda_{\tilde{H}_{\text{rot}}}^2, \bar{\lambda}_{\tilde{H}_{\text{rot}}}^1\rangle, |\lambda_{\tilde{H}_{\text{rot}}}^2, \bar{\lambda}_{\tilde{H}_{\text{rot}}}^2\rangle$ are the eigenvectors of the increased evolution operator $S_{t,\omega}$ for arbitrary $\omega \in (0, 1]$. Corresponding eigenvalues are respectively $0, -2i\sqrt{3}\omega, 2i\sqrt{3}\omega, 0$. Similarly to the example presented in the previous section, the state

$$\tilde{\varrho}(0) = \frac{1}{2}(|\lambda_{\tilde{H}_{\text{rot}}}^1\rangle + |\lambda_{\tilde{H}_{\text{rot}}}^2\rangle)(\langle\lambda_{\tilde{H}_{\text{rot}}}^1| + \langle\lambda_{\tilde{H}_{\text{rot}}}^2|) \quad (16)$$

is the required initial state. The state after time t takes the form

$$\tilde{\varrho}(t) = \frac{1}{2}(|\lambda_{\tilde{H}_{\text{rot}}}^1\rangle\langle\lambda_{\tilde{H}_{\text{rot}}}^1| + e^{-2it\sqrt{3}\omega}|\lambda_{\tilde{H}_{\text{rot}}}^1\rangle\langle\lambda_{\tilde{H}_{\text{rot}}}^2| + e^{2it\sqrt{3}\omega}|\lambda_{\tilde{H}_{\text{rot}}}^2\rangle\langle\lambda_{\tilde{H}_{\text{rot}}}^1| + |\lambda_{\tilde{H}_{\text{rot}}}^2\rangle\langle\lambda_{\tilde{H}_{\text{rot}}}^2|). \quad (17)$$

It is easy to see that the $\tilde{\varrho}(t)$ is periodic with period $\frac{\pi}{\sqrt{3}\omega}$. \square

One can note, that the probability distribution obtained from the measurement in the basis of the system changes in time. However, when we analyse the canonical measurement from [14], where as measurement operators we choose the projections onto the subspaces of corresponding to different vertices, we can observe that in the following example the probability does not change. In this case there is probability one of measuring vertex v_0 for each time point. In fact, we haven't found any combination of a graph and an initial state for which the probability distribution does not converge. Hence we propose the following conjecture.

Conjecture. *Let us choose the nonmoralizing evolution model. Let $\Pi(\varrho(0), t)$ denotes the probability distribution of canonical measurement onto subspaces of vertices in time t with initial state $\varrho(0)$. Then for arbitrary $\varrho(0)$ there exists probability distribution Π_∞ such that*

$$\lim_{t \rightarrow \infty} \Pi(\varrho(0), t) = \Pi_\infty. \quad (18)$$

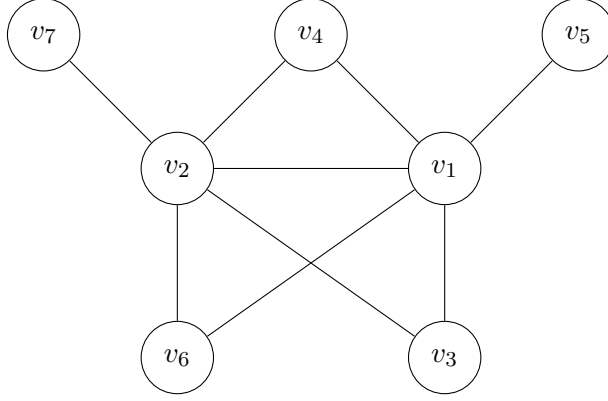


Figure 3: Graph for which there exists two different stationary states in the sense of the natural measurement. The states can be obtained by starting in vertices v_7 and v_6

Note, that the probability distribution may be nonunique. To see that let us analyse the graph presented in Fig. 3. We choose $\omega = \frac{1}{2}$ and two initial states $\tilde{\varrho}_0 = |v_6^0\rangle\langle v_6^0|$ and $\tilde{\varrho}_1 = |v_7^0\rangle\langle v_7^0|$. We have found the limiting probability distribution

$$\Pi_0 := \lim_{t \rightarrow \infty} \Pi(\varrho_0, t), \quad (19)$$

$$\Pi_1 := \lim_{t \rightarrow \infty} \Pi(\varrho_1, t). \quad (20)$$

The probability distributions differs, for example $\Pi_0(v_0) = 0.666616$ and $\Pi_1(v_0) = 0.11897$.

4 The digraph structure observance

In [14] it was suggested, that the global interaction quantum stochastic walk can be applied for modelling the walk on directed graphs. Moreover, an example suggesting that the original global interaction evolution, where Lindbladian operator is an adjacency matrix of the directed graph, does not preserve the digraph structure have been provided. Let us take the graph from Fig. 4(a) and let us analyse the graph without the Hamiltonian. One can find that

$$\frac{1}{4}(|v_1\rangle - |v_2\rangle)(\langle v_1| - \langle v_2|) + \frac{1}{2}|v_3\rangle\langle v_3| \quad (21)$$

is proper stationary state. There is a nonzero probability of measuring the state in vertex v_1 and v_2 .

Evolution on directed graph, at least for Lindbladian-part evolution only, should have the following property. If for an arbitrary vertex there is path to some sink vertex, then an arbitrary initial state ϱ should converge to the state spanned by vectors corresponding to the sink vertices. In the case of our example arbitrary state should converge to the state $|v_3\rangle\langle v_3|$.

The unintuitive stationary state (21) comes from the spontaneous moralization of the graph [14]. Since the moralization of the graph was corrected, it is necessary to check whether the improper stationary state still occurs in the nonmoralizing evolution. In this section we obtain the results by analysing the stationary states.

The numerical analysis was performed as follows. Let S be a set of all sink vertices. We start in some vertex with nonzero outdegree. We determine the state ϱ_∞ for large time value and we verified numerically, whether it is close to stationary state in the sense of probability distribution of measurement. Then we compute the cumulated probability of measuring the state in the sink vertices

$$p_S(\varrho_\infty) = \sum_{v \in S} \langle v | \varrho_\infty | v \rangle \quad (22)$$

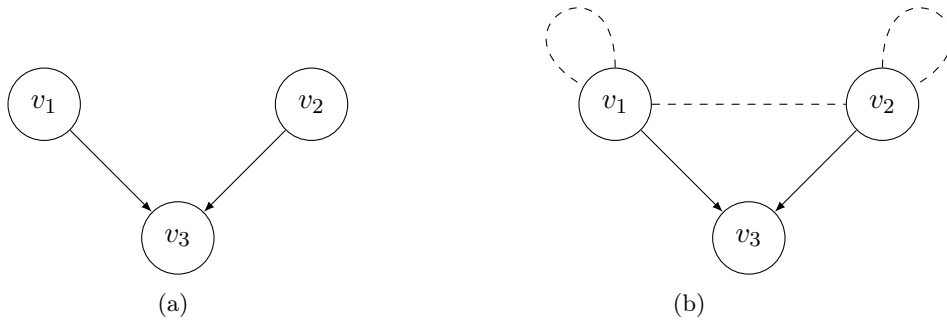


Figure 4: Visualisation of a directed graph (a) and its spontaneous moralization (b)

and the second moment of the distance from the sink vertices

$$\mu_S(\varrho_\infty) = \sum_{v \in V} d^2(v, S) \langle v | \varrho_\infty | v \rangle, \quad (23)$$

where $d(v, S)$ is the length of the shortest path from v to closest sink vertex. We say, that the greater the value of p_{sink} and the lower the value of μ_{sink} are, the more the evolution preserves the graph. We have analysed the evolution for $\omega \in (0, 1]$. For the purpose of our analysis we have selected four types of graphs, namely: a line segment, the Peterson graph, the Apollonian graph, and the Sierpinski triangle. We choose the orientation of the graphs such that each vertex is either a sink vertex, or there is a path from it to some sink vertex.

The obtained results are presented in Fig. 5. One can see that the larger the value of ω is, the more probability cumulates in the close neighbourhood of the sink vertices. Moreover we claim that for $\omega \in [0.7, 1]$ values of p_{sink} and μ_{sink} grows with ω . One can also notice, that in the $\omega \rightarrow 1$ limit the p_{sink} converge to one and μ_{sink} vanishes. The above analysis suggests, that when $\omega = 0$, the directed graph structure tends to is fully preserved.

5 Conclusions

In this paper we considered a limiting properties of quantum stochastic walk with global interaction. We have proved that in the case of moralizing evolution, on an undirected graph the stationary states of the evolution are precisely those from original continuous quantum walk.

Contrary we provide an example of directed graph, for which there exists not-converging initial state. Furthermore, we present similar counterexample for the case of non-moralizing evolution. In our example the evolution is periodic, and at the same time, the probability of measurement of the subspace corresponding to vertices do not change in time. Numerical results suggest that the property holds for an arbitrary configuration.

We utilize the stationary states to argue that the nonmoralizing evolution preserves the structure of the digraph. The arguments are based on the numerical analysis the total probability in the sink vertices and the second moment of distance from the nearest sink vertices for selected types of graphs. The result suggest that there exists a bound for the smoothing parameter above which the analysed quantities grow monotonically.

Acknowledgements

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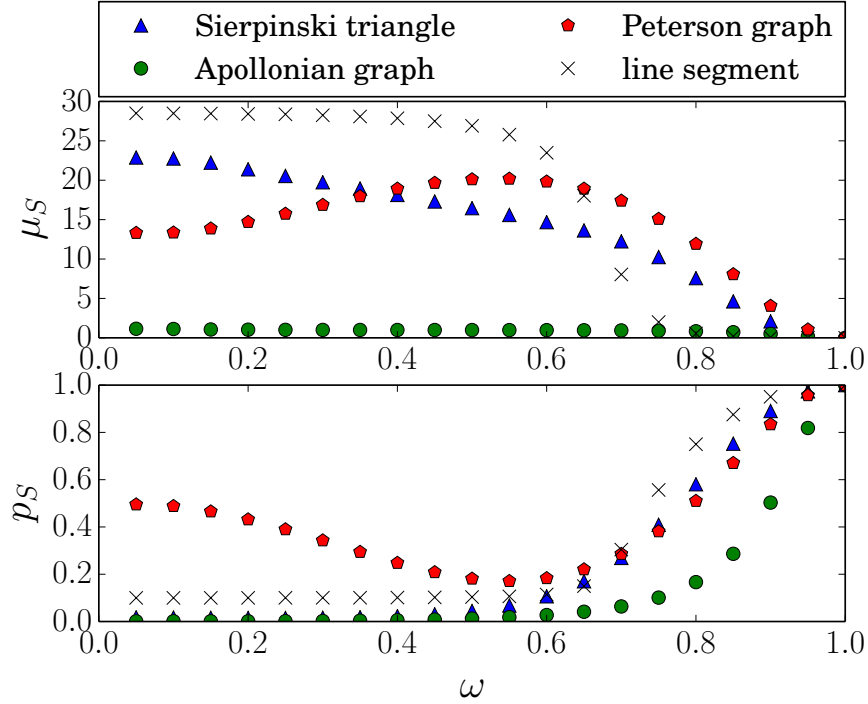


Figure 5: Top plot shows the second moment of the distance from the vertex to the closest sink vertex. Bottom plot shows the probability of measuring one of the sink vertex for large time graphs. Sierpinski triangle has 15 vertices, Apollonian graph has 12 vertices and length of line segment equals 10.

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